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Quantum Structures: An Attempt to Explain the Origin of Their Appearance in Nature

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We explain quantum structure as due to two effects: (a) a real change of state of the entity under the influence of the measurement and (b) a lack of knowledge about a deeper deterministic reality of the measurement process. We present a quantum machine, with which we can illustrate in a simple way how the quantum structure arises as a consequence of the two mentioned effects. We introduce a parameter ϵ that measures the size of the lack of knowledge of the measurement process, and by varying this parameter, we describe a continuous evolution from a quantum structure (maximal lack of knowledge) to a classical structure (zero lack of knowledge). We show that for intermediate values of ϵ we find a new type of structure that is neither quantum nor classical. We apply the model to situations of lack of knowledge about the measurement process appearing in other aspects of reality. Specifically, we investigate the quantumlike structures that appear in the situation of psychological decision processes, where the subject is influenced during the testing and forms some opinions during the testing process. Our conclusion is that in the light of this explanation, the quantum probabilities are epistemic and not ontological, which means that quantum mechanics is compatible with a determinism of the whole.

1. A MACROSCOPIC MACHINE PRODUCING QUANTUM STRUCTURE

Before we identify the origin of the appearance of quantum structures in nature, we describe a macroscopic machine that produces quantum structure. We shall make use intensively of the internal functioning of this machine to demonstrate our general explanation.

The machine that we consider consists of a physical entity S that is a point particle P that can move on the surface of a sphere, denoted *surf*, with center O and radius 1. The unit vector v where the particle is located on *surf*

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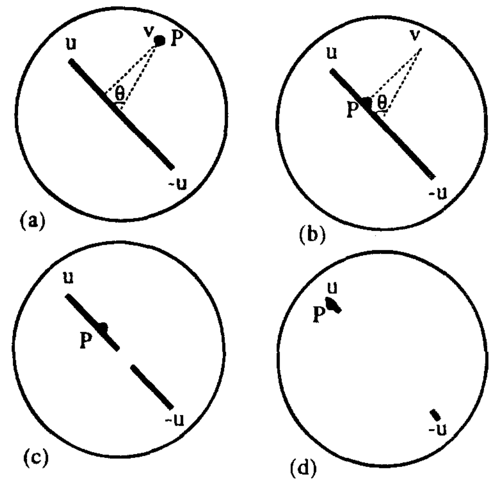


Fig. 1. A representation of the quantum machine. (a) The physical entity P is in state p_v in the point v, and the elastic corresponding to the experiment e_u is installed between the two diametrically opposed points u and $-u$. (b) The particle P falls orthogonally onto the elastic and sticks to it. (c) The elastic breaks and the particle P is pulled toward the point u, such that (d) it arrives at the point u, and the experiment e_u gets the outcome o_1^u.

represents the state p_v of the particle (see Fig. 1a). For each point $u \in surf$, we introduce the following experiment e_u. We consider the diametrically opposite point $-u$, and install a piece of elastic of length 2 such that it is fixed with one of its endpoints in u and the other endpoint in $-u$. Once the elastic is installed, the particle P falls from its original place v orthogonally onto the elastic, and sticks on it (Fig. 1b). Then the elastic breaks and the particle P, attached to one of the two pieces of the elastic (Fig. 1c), moves to one of the two endpoints u or $-u$ (Fig. 1d). Depending on whether the particle P arrives at u (as in Fig. 1) or at $-u$, we give the outcome o_1^u or o_2^u to e_u. In Fig. 2 we represent the experimental process connected to e_u in the plane where it takes place, and we can easily calculate the probabilities

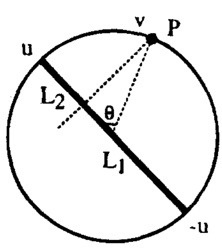


Fig. 2. A representation of the experimental process in the plane where it takes place. The elastic of length 2, corresponding to the experiment e_u, is installed between u and $-u$. The probability $P(o_1^u | p_v)$ that the particle P ends up in point u is given by the length of the piece of elastic L_1 divided by the length of the total elastic. The probability $P(o_2^u | p_v)$ that the particle P ends up in point $-u$ is given by the length of the piece of elastic L_2 divided by the length of the total elastic.

corresponding to the two possible outcomes. In order to do so we remark that the particle P arrives at u when the elastic breaks at a point of the interval L_1, and arrives at $-u$ when it breaks at a point of the interval L_2 (see Fig. 2). We make the hypothesis that the elastic breaks uniformly, which means that the probability that the particle, being in state p_v, arrives at u, is given by the length of L_1 (which is $1 + \cos\theta$) divided by the length of the total elastic (which is 2). The probability that the particle in state p_v arrives at $-u$ is the length of L_2 (which is $1 - \cos\theta$) divided by the length of the total elastic. If we denote these probabilities respectively by $P(o_1^u, p_v)$ and $P(o_2^u, p_v)$, we have

$$P(o_1^u, p_v) = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2} \tag{1}$$

$$P(o_2^u, p_v) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{2}$$

The probabilities that we find in this way are exactly the quantum probabilities for the spin measurement of a spin-1/2 quantum entity, which means that we can describe this macroscopic machine by the ordinary quantum formalism with a two-dimensional complex Hilbert space as the carrier for the set of states of the entity.

2. QUANTUM STRUCTURES

Already from the advent of quantum mechanics it was known that the structure of quantum theory is very different from the structure of classical theories. This structural difference has been expressed and studied in terms of different mathematical categories, and we mention here some of the most important ones: (1) if one considers the collection of properties (experimental propositions) of a physical entity, then it has the structure of a Boolean lattice for the case of a classical entity, while it is non-Boolean for the case of a quantum entity (Birkhoff and von Neumann, 1936; Jauch, 1968; Piron, 1976), (2) for the probability model, it can be shown that for a classical entity it is Kolmogorovian, while for a quantum entity it is not (Foulis and Randall, 1971; Randall and Foulis, 1979, 1983; Gudder, 1988; Accardi, 1982; Pitovski, 1989), (3) if the collection of observables is considered, a classical entity gives rise to a commutative algebra, while a quantum entity not (Segal, 1947; Emch, 1984).

The presence of these deep structural differences between classical theories and quantum theory has contributed strongly to the belief that classical theories describe the ordinary "understandable" part of reality, while quantum theory confronts us with a part of reality (the microworld) that is impossible to understand. Therefore there is still the strong belief that *quantum mechanics cannot be understood.* The example of our macroscopic machine with a quantum structure challenges this, because obviously the functioning of this machine can be understood. The aim of this paper is to show that the main part of quantum structures can indeed be explained in this way and that the reason why they appear in nature can be identified. In this paper we shall analyze this explanation, which we have named the "hidden measurement approach," within the category of the probability models. We refer to Aerts and Van Bogaert (1992), Aerts *et al.* (1993a,b), Aerts (1994), and Aerts and Durt (1994a,b) for an analysis of this explanation in terms of other categories.

The original development of probability theory aimed at a formalization of the description of a probability that appears as the consequence of *a lack of knowledge*. The probability structure appearing in situations of lack of knowledge was axiomatized by Kolmogorov and such a probability model is now called Kolmogorovian. Since the quantum probability model is not Kolmogorovian, it has now generally been accepted that the quantum probabilities are *not* a description of a *lack of knowledge*. Sometimes this conclusion is formulated by stating that the quantum probabilities are *ontological* probabilities, as if they would be present in reality itself. In the hidden measurement approach we show that the quantum probabilities can be explained as being due to a *lack of knowledge*, and we prove that what distinguished quantum probabilities from classical Kolmogorovian probabilities is the *nature of this lack of knowledge*. Let us go back to the quantum machine to illustrate what we mean.

If we consider again our quantum machine (Figs. 1 and 2) and look for the origin of the probabilities as they appear in this example, we can remark that the probability is entirely due to a *lack of knowledge* about the measurement process, namely, the lack of knowledge of where exactly the elastic breaks during a measurement. More specifically, we can identify two main aspects of the experiment e_u as it appears in the quantum machine:

- The experiment e_u effects a real change on the state p_v of the entity S. Indeed, the state p_v changes into one of the states p_u or p_{-u} by the experiment e_u.
- The probabilities appearing are due to a *lack of knowledge* about a deeper reality of the individual measurement process itself, namely where the elastic breaks.

These two effects give rise to quantumlike structures. The lack of knowledge about a deeper reality of the individual measurement process we have referred to as the presence of "hidden measurements" that operate deterministically in this deeper reality (Aerts, 1986, 1987, 1991), and this is the origin of the name that we gave to this approach. A consequence of this explanation is that quantum structures turn out to be present in many other aspects of reality where the two mentioned effects appear. We think of the many situations in the human sciences where generally the measurement disturbs profoundly the entity under study, and where there is almost always a lack of knowledge about the deeper reality of what is going on during this measurement process. In the final part of this paper we give some examples of quantum structures appearing in such situations.

3. QUANTUM, CLASSICAL, AND INTERMEDIATE

If the quantum structure can be explained by a lack of knowledge on the measurement process, we can go a step further, and wonder what types of structure arise when we consider the original models, with lack of knowledge on the measurement process, and introduce a variation of the magnitude of this lack of knowledge. We have studied the quantum machine under varying "lack of knowledge," parametrizing this variation by a number $\epsilon \in [0, 1]$, such that $\epsilon = 1$ corresponds to the situation of maximal lack of knowledge, giving rise to a quantum structure, and $\epsilon = 0$ corresponds to the situation of zero lack of knowledge, generating a classical structure, and other values of ϵ correspond to intermediate situations, giving rise to a structure that is neither quantum nor classical (Aerts *et al.*, 1992, 1993a). It is this model that we have called the ϵ-model, and we want to introduce it again in this paper.

We start from the quantum machine, but introduce different types of elastic. An ϵ, d-elastic consists of three different parts: one lower part, where it is unbreakable, a middle part, where it breaks uniformly, and an upper part, where it is again unbreakable. By means of the two parameters $\epsilon \in [0, 1]$ and $d \in [-1 + \epsilon, 1 - \epsilon]$, we fix the sizes of the three parts in the following way. Suppose that we have installed the ϵ, d-elastic between the points $-u$ and u of the sphere. Then the elastic is unbreakable in the lower part from $-u$ to $(d - \epsilon)\cdot u$, it breaks uniformly in the part from $(d - \epsilon)\cdot u$ to $(d + \epsilon) \cdot u$, and it is again unbreakable in the upper part from $(d + \epsilon)\cdot u$ to u (see Fig. 3).

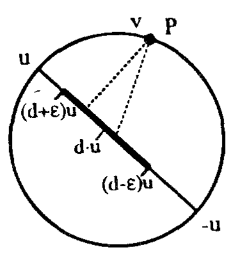


Fig. 3. A representation of the experiment $e^{\epsilon}_{u,d}$. The elastic breaks uniformly between the points $(d - \epsilon)u$ and $(d + \epsilon)u$, and is unbreakable in other points.

An e_u experiment performed by means of an ϵ, d-elastic shall be denoted by $e^\epsilon_{u,d}$. We have the following cases:

1. $v \cdot u \leq d - \epsilon$. The particle sticks to the lower part of the ϵ, d-elastic, and any breaking of the elastic pulls it down to the point $-u$. We have $P^\epsilon(o^u_1, p_v) = 0$ and $P^\epsilon(o^u_2, p_v) = 1$.

2. $d - \epsilon < v \cdot u < d + \epsilon$. The particle falls onto the breakable part of the ϵ, d-elastic. We can easily calculate the transition probabilities and find

$$P^\epsilon(o^u_1, p_v) = \frac{1}{2\epsilon}(v \cdot u - d + \epsilon) \tag{3}$$

$$P^\epsilon(o^u_2, p_v) = \frac{1}{2\epsilon}(d + \epsilon - v \cdot u) \tag{4}$$

3. $d + \epsilon \leq v \cdot u$. The particle falls onto the upper part of the ϵ, d-elastic, and any breaking of the elastic pulls it upward, such that it arrives in u. We have $P^\epsilon(o^u_1, p_v) = 1$ and $P^\epsilon(o^u_2, p_v) = 0$.

4. PROBABILITIES APPEARING IN PHYSICAL SITUATIONS

If we want to analyze the structure of the quantum probability model in the light of axioms that have been formulated for classical probability theory, we first have to be very specific about the situation that we consider. In physics (and hence also in quantum mechanics and classical mechanics) we consider a situation where we have a physical entity S that can be in different states $p, q, r, \ldots$, and we will denote the set of states by Σ. On this physical entity S, in a certain state p, we perform experiments $e, f, g, \ldots$, that respectively have sets of possible outcomes $O_e, O_f, O_g, \ldots$. Let us denote the collection of all relevant experiments by $\mathscr{E}$. There are different places in which probability appears in this scheme.

1. *The probability connected to the states.* In many occasions it is not possible to prepare the entity S in such a way that we know in which states it is before we start an experiment. We can only prepare it such that we are left with a situation of "lack of knowledge" about the state of the entity. This situation of lack of knowledge is described by means of a probability measure $\mu: \mathscr{B}(\Sigma) \to [0, 1]$ on the set of states, such that $\mathscr{B}(\Sigma)$ is a σ-algebra of measurable subsets of Σ, and for $K \in \mathscr{B}(\Sigma)$ we have that $\mu(K)$ is the probability that the state of the entity S is in the subset K. We have (a) $\mu(\Sigma) = 1$, and (b) $\mu(\cup_i K_i) = \Sigma_i \mu(K_i)$ for sets K_i such that $K_n \cap K_m = \emptyset$ for $n \neq m$. What we call "states" $p, q, r, \ldots$ are often called "pure states" and what we call "situations of lack of knowledge on the states" $\mu, \nu, \ldots$ are often called "mixed states."

2. *The probability connected to the experiments.* Even when the entity S is in a state p and an experiment e is performed, probability, defined as the limit of the relative frequency connected to an outcome $o_k \in O_e$, appears. For a fixed state p, the probability that an experiment e gives an outcome in a subset $A^e \subset O_e$, denoted by $P(A^e, p)$, can be described as a probability measure on the outcome set O_e of the experiment e. Hence a map P: $\mathcal{B}(O_e) \times \Sigma \rightarrow [0, 1]$ such that $\mathcal{B}(O_e)$ is a collection of measurable subsets of O_e and (a) $P(O_e, p) = 1$, and (b) $P(\cup A_k^e, p) = \Sigma_k P(A_k^e, p)$ for sets A_k^e such that $A_i^e \cap A_j^e = \emptyset$ for $i \neq j$.

3. *The general probability.* Most of the time we measure a probability in the laboratory that contains both just mentioned probabilities. It is the probability that in a situation μ of lack of knowledge on the states, an experiment e gives an outcome in a subset $A^e \subset O_e$, and we will denote it by $P(A^e, \mu)$. This probability is, for a given experiment e, a map P: $\mathcal{B}(O_e) \times \mathcal{M}(\Sigma) \rightarrow [0, 1]$, where $\mathcal{M}(\Sigma)$ is the set of probability measures on Σ.

4. *The eigenstate sets and the possibility-state sets.* As in Aerts (1994), we introduce for an experiment e the eigenstate sets as maps eig: $\mathcal{P}(O_e) \rightarrow \mathcal{P}(\Sigma)$, where for $A^e \subset O_e$ we have

$$eig(A^e) = \{p \mid p \in \Sigma, \text{ if } S \text{ is in } p, \text{ then the outcome of } e \text{ occurs with certainty in } A^e\} \quad (5)$$

We also introduce the possibility-state sets as maps pos: $\mathcal{P}(O_e) \rightarrow \mathcal{P}(\Sigma)$, where for $A^e \subset O_e$ we have

$$pos(A^e) = \{p \mid p \in \Sigma, \text{ if } S \text{ is in } p, \text{ then the outcome of } e \text{ occurs possibly in } A^e\} \quad (6)$$

Clearly we always have

$$eig(A^e) \subset pos(A^e) \quad (7)$$

Theorem 1. We consider an entity S in a situation with lack of knowledge about the states described by the probability measure μ on the state space. For an arbitrary experiment e and set of outcomes $A^e \subset O_e$ we have

$$\mu(eig(A^e)) \leq P(A^e, \mu) \leq \mu(pos(A^e))\ ^2 \quad (8)$$

Proof. As defined, we have that $\mu(eig(A^e))$ is the probability that the state of the entity is in the subset $eig(A^e)$. If the state is in $eig(A^e)$, the experiment e gives with certainty an outcome in A^e, and therefore $\mu(eig(A^e))$

[2] In case $eig(A^e)$ and/or $pos(A^e)$ are nonmeasurable subsets, we use the outer measure of μ for $eig(A^e)$ and the inner measure of μ for $pos(A^e)$.

$\leq P(A^e, \mu)$. As defined, we have that $\mu(pos(A^e))$ is the probability that the state of the entity is in $pos(A^e)$. If the state is in $pos(A^e)$, the experiment e has a possible outcome in A^e, and therefore $P(A^e, \mu) \leq \mu(pos(A^e))$.

We shall show in the next section that for a classical probability model, the two inequalities become equalities. But first we want to illustrate all these concepts on the quantum machine.

5. ILLUSTRATION ON THE QUANTUM MACHINE

It is easy to see how these concepts are defined for the quantum machine (and also for the ϵ-model). For a considered experiment e_u (or $e^{\epsilon}_{u,d}$ in the ϵ-model), we have an outcome set $O_u = \{o^u_1, o^u_2\}$. The set of states is $\Sigma = \{p_v \mid v \in surf\}$. The situations with lack of knowledge about the states are described by probability measures on the surface of the sphere. We have also described $P^{\epsilon}(A_u, p_v)$ for an arbitrary $A_u \subset O_u$ [see (3) and (4)]. For the eigenstate sets and the possibility-state sets we have (see Fig. 4)

$$eig^{\epsilon}(\{o^u_1\}) = \{p_v \mid d + \epsilon \leq v \cdot u\}, \qquad eig^{\epsilon}(\{o^u_2\}) = \{p_v \mid v \cdot u \leq d - \epsilon\} \quad (9)$$

$$pos^{\epsilon}(\{o^u_1\}) = \{p_v \mid d - \epsilon < v \cdot u\}, \qquad pos^{\epsilon}(\{o^u_2\}) = \{p_v \mid v \cdot u < d + \epsilon\} \quad (10)$$

We can consider the following specific cases:

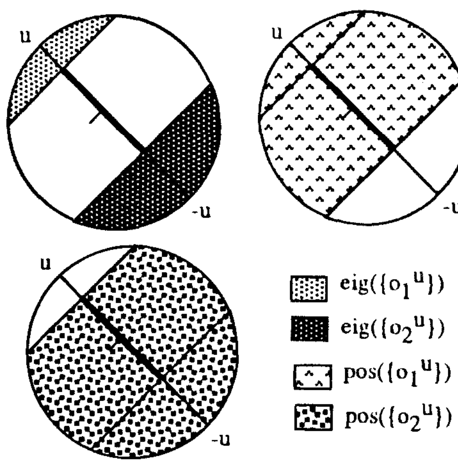


Fig. 4. The eigenstate sets $eig^{\epsilon}(\{o^u_1\})$ and $eig^{\epsilon}(\{o^u_2\})$. If the state of the entity (the position of the particle P) is in $eig^{\epsilon}(\{o^u_1\})$ [or in $eig^{\epsilon}(\{o^u_2\})$], then the experiment $e^{\epsilon}_{u,d}$ gives with certainty the outcome o^u_1 [or with certainty the outcome o^u_2]. We also have represented the possibility state sets, $pos^{\epsilon}(\{o^u_1\})$ [$pos^{\epsilon}(\{o^u_2\})$], the collection of states where the entity gives a possible outcome o^u_1 [o^u_2].

1. *The quantum situation* ($\epsilon = 1$). For $\epsilon = 1$ we always have $d = 0$, and the ϵ-model reduces to the original quantum machine that we introduced in Section 1. It is a model for the spin of a spin-1/2 quantum entity. The transition probabilities are the same as the ones related to the outcomes of a Stern–Gerlach spin experiment on a spin-1/2 quantum particle, of which the quantum spin state in direction $v = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, denoted by ψ_v, and the experiment e_u corresponding to the spin experiment in direction $u = (\cos\beta \sin\alpha, \sin\beta \sin\alpha, \cos\alpha)$, are described respectively by the vector and the self-adjoint operator

$$\psi_v = (e^{-i\phi/2}\cos\theta/2, e^{i\phi/2}\sin\theta/2), \qquad H_u = \frac{1}{2}\begin{pmatrix} \cos\alpha & e^{-i\beta}\sin\alpha \\ e^{i\beta}\sin\alpha & -\cos\alpha \end{pmatrix} \tag{11}$$

of a two-dimensional complex Hilbert space. For the eigenstate sets and possibility states sets we find

$$eig^{\epsilon}(\{o_1^u\}) = \{p_u\}, \qquad eig^{\epsilon}(\{o_2^u\}) = \{p_{-u}\}$$
$$pos^{\epsilon}(\{o_1^u\}) = \{p_v | v \neq -u\}, \qquad pos^{\epsilon}(\{o_2^u\}) = \{p_v | v \neq u\} \tag{12}$$

Suppose that we consider a situation with lack of knowledge about the state, described by a uniform probability distribution μ on the sphere, which corresponds to a random distribution of the point on the sphere. Then we can easily calculate the probabilities

$$P^{\epsilon}(\{o_1^u\}, \mu) = \frac{1}{2}, \qquad P^{\epsilon}(\{o_1^u\}, \mu) = \frac{1}{2} \tag{13}$$

On the other hand, we have

$$\mu(eig^{\epsilon}(\{o_1^u\})) = 0, \qquad \mu(eig^{\epsilon}(\{o_2^u\})) = 0$$
$$\mu(pos^{\epsilon}(\{o_1^u\})) = 1, \qquad \mu(pos^{\epsilon}(\{o_2^u\})) = 1 \tag{14}$$

which shows that the inequalities of Theorem 1 [see (8)] are very strong in this quantum case.

2. *The classical situation* ($\epsilon = 0$). The classical situation is the situation without fluctuations. If $\epsilon = 0$, then d can take any value in the interval $[-1, +1]$, and we have

$$eig^{\epsilon}(\{o_1^u\}) = \{p_v | d < v \cdot u\}, \qquad eig^{\epsilon}(\{o_2^u\}) = \{p_v | v \cdot u < d\} \tag{15}$$

$$pos^{\epsilon}(\{o_1^u\}) = \{p_v | d \leq v \cdot u\}, \qquad pos^{\epsilon}(\{o_2^u\}) = \{p_v | v \cdot u \leq d\} \tag{16}$$

We again consider the situation of a random distribution of the point particle on the sphere described by the probability distribution μ. We then have in this case

$$P^{\epsilon}(\{o_1^u\}, \mu) = \frac{1}{2}(1 - d), \qquad P^{\epsilon}(\{o_2^u\}, \mu) = \frac{1}{2}(1 + d) \tag{17}$$

On the other hand, we have

$$\mu(eig^{\epsilon}(\{o_1^u\})) = \frac{1}{2}(1 - d) = \mu(pos^{\epsilon}(\{o_1^u\}))$$

$$\mu(eig^{\epsilon}(\{o_2^u\})) = \frac{1}{2}(1 + d) = \mu(pos^{\epsilon}(\{o_2^u\})) \tag{18}$$

which shows that the inequalities of Theorem 1 [see (8)] have become equalities in this case.

3. *The general situation.* To give a clear picture of the general situation, we introduce additional concepts. First we remark that the regions of eigenstates $eig^{\epsilon}(\{o_1^u\})$ and $eig^{\epsilon}(\{o_2^u\})$ and the regions of possibility states $pos^{\epsilon}(\{u_1^u\})$ and $pos^{\epsilon}(\{o_2^u\})$ are determined by the points of spherical sectors of *surf* centered around the points u and $-u$ (see Figs. 4 and 5). We denote a closed spherical sector centered around the point $u \in surf$ with angle θ by $sec(u, \theta)$, and an open spherical sector with the same angle by $sec^o(u, \theta)$. We call λ_d^{ϵ} the angle of the spherical sectors corresponding to $eig^{\epsilon}(\{o_1^u\})$ for all u; hence for $0 \neq \epsilon$ we have $eig^{\epsilon}(\{o_1^u\}) = \{p_v | v \in sec(u, \lambda_d^{\epsilon})\}$, and $eig^{\epsilon}(\{u_1^o\}) = \{p_v | v \in sec^o(u, \lambda_d^0)\}$ for $\epsilon = 0$ (see Fig. 5). We can verify easily that $eig^{\epsilon}(\{o_2^u\})$ is determined by a spherical sector centered around the point $-u$. We call μ_d^{ϵ} the angle of this spherical sector; hence, for $0 \neq \epsilon$ we have $eig^{\epsilon}(\{o_2^u\}) = \{p_v | v \in sec(-u, \mu_d^{\epsilon})\}$, and $eig^{\epsilon}(\{o_2^u\}) = \{p_v | v \in sec^o(-u, \mu_d^{\epsilon})\}$ for $\epsilon = 0$. For $0 \neq \epsilon$ we have $pos^{\epsilon}(\{o_1^u\}) = \{p_v | v \in sec^o(u, \pi - \mu_d^{\epsilon})\}$, and $pos^{\epsilon}(\{o_1^u\}) = \{p_v | v \in sec(-u, \pi - \mu_d^0)\}$ for $\epsilon = 0$.

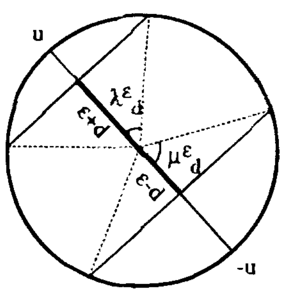


Fig. 5. The different angles λ_d^{ϵ} and μ_d^{ϵ} characterizing the spherical sectors of the eigenstate sets and possibility state sets.

For $0 \neq \epsilon$ we have $pos(\{o_2^u\}) = \{p_v | v \in sec^o(-u, \pi - \lambda_d^\epsilon)\}$, and $pos^\epsilon(\{o_2^u\}) = \{p_v | v \in sec(-u, \pi - \lambda_d^\epsilon)\}$ for $\epsilon = 0$. We have

$$\cos \lambda_d^\epsilon = \epsilon + d, \qquad \cos \mu_d^\epsilon = \epsilon - d, \qquad \lambda_{-d}^\epsilon = \mu_d^\epsilon \tag{19}$$

6. THE CLASSICAL SITUATION

We want to formulate now the classical situation in this general scheme. We shall see that in the characterization of a classical probability model, the inequalities of Theorem 1 play an important role.

Definition 1. Suppose that we are in the situation μ of lack of knowledge on the states. We say that an experiment e is a "classical experiment" iff $\mu(eig(A^e)) = \mu(pos(A^e))$ for all subsets $A^e \subset O_e$. In other words, a classical experiment is an experiment where all states, except a collection of measure zero, give rise to predetermined outcomes for this experiment.

Classical experiments are experiments with predetermined outcomes (except for a set of states of measure zero that as a consequence do not contribute to the statistics). For these classical experiments we can show that probability always originates in a lack of knowledge on the states.

Theorem 2. If we are in the situation μ of lack of knowledge on the states, and e is a classical experiment, then for each $A^e \in \mathcal{B}(O_e)$ we have

$$\mu(eig(A^e)) = P(A^e, \mu) = \mu(pos(A^e)) \tag{20}$$

Proof. A direct consequence of Definition 1 and Theorem 1.

We shall show now that for classical experiments Bayes' formula for the conditional probability is valid. To be able to analyze the validity of Bayes' formula in the scheme that we have presented here, we must give an operational definition for the concept of conditional probability. Here we are confronted with a conceptual problem, since in most textbooks, the conditional probability is defined by means of Bayes' formula. Since the conditional probability is a primary physical quantity that is measured in the laboratory, we should define it operationally and without the use of Bayes' formula.

7. THE CONCEPT OF CONDITIONAL PROBABILITY

We want to make clear that there is a distinction between the occurrence of an outcome when an experiment is performed, and the conditioning on an outcome corresponding to an experiment.

Definition 2. Given a situation μ of lack of knowledge on the states of an entity S, described by a probability measure on this set of states Σ, we condition the entity S on a subset $A_f \subset O_f$ for an experiment f if we consider during the performance of the experiment e only those trials where the situation of the entity before the experiment e is such that we can predict the outcome for the experiment f to occur with certainty in A_f, if we decided to perform the experiment f.

From this definition it follows that conditioning is equivalent to a change of the situation μ before the experiment in such a way that the experiment f would give with certainty an outcome in A_f, if it were to be executed. The new situation of lack of knowledge is described by the probability measure that we shall denote by μ_{A_f}: $\mathcal{B}(\Sigma) \to [0, 1]$. It is defined for an arbitrary subset $K \subset \Sigma$ as follows:

$$\mu_{A_f}(K) = \mu(K \cap eig(A_f))/\mu(eig(A_f)) \tag{21}$$

Now that we have introduced this concept of "conditioning" on an experiment, we can introduce the general concept of conditional probability.

Definition 3. Given a situation μ of lack of knowledge on the states of an entity, described by the probability measure μ, and given two experiments e and f, then we want to consider the conditional probability $P(A_e, A_f, \mu)$. This is the probability that the experiment e makes occur an outcome in the set A_e when the situation is conditioned on the set A_f for the experiment f. The conditional probability is a map P: $\mathcal{B}(O_e) \times \mathcal{B}(O_f) \times \mathcal{M}(\Sigma) \to [0, 1]$.

Theorem 3. Given a situation μ of lack of knowledge on the states, and two experiments e and f, if the experiment e is a classical experiment, then the conditional probability $P(A_e, A_f, \mu)$ satisfies Bayes' formula. More specifically, we have

$$P(A_e, A_f, \mu) = \mu(eig(A_e) \cap eig(A_f))/\mu(eig(A_f))$$

Proof. We have [see (21)] that $P(A_e, A_f, \mu) = P(A_e, \mu_{A_f})$. If e is a classical experiment, it follows from Theorem 2 that $P(A_e, \mu_{A_f}) = \mu_{A_f}(eig(A_e)) = \mu(eig(A_e) \cap eig(A_f))/\mu(eig(A_f))$. This shows that Bayes' formula is valid.

From this theorem it can intuitively be seen that Bayes' formula for the conditional probability is not valid for the case of experiments that are nonclassical. We shall now analyze all these situations in the ϵ-model and show how the conditional probability in the ϵ-model evolves continuously from the quantum transition probability, for the case of $\epsilon = 1$, to a classical Kolmogorovian probability satisfying Bayes' formula, for the case $\epsilon = 0$. We shall also show that for values of ϵ strictly between 1 and 0, the conditional probability is neither quantum nor classical.

8. THE CONDITIONAL PROBABILITY AND THE ϵ-MODEL

Given a situation μ of lack of knowledge about the state of the point particle described by a uniform probability measure on the sphere, this corresponds to the situation where the particle P is distributed at random on the sphere. For a fixed ϵ, and two parameters d and c both in the interval $[-1 + \epsilon, 1 - \epsilon]$, there are also given the two experiments $e^{\epsilon}_{u,d}$ and $e^{\epsilon}_{w,c}$ (Fig. 6). In general we consider the conditional probability for arbitrary elements of the set of measurable subsets of the outcome sets of the two experiments. Since in the ϵ-model we only have experiments $e^{\epsilon}_{u,d}$, with two outcomes o^u_1 and o^u_2, we want to alleviate somewhat the notation. Therefore we shall denote the conditional probability that the experiment $e^{\epsilon}_{u,d}$ gives the outcome o^u_1 (respectively o^u_2), when the entity is conditioned for the outcome o^w_1 of the experiment $e^{\epsilon}_{w,c}$, by $P(u, w, \mu)$ [respectively $P(-u, w, \mu)$]. In a similar way we denote the conditional probability that the experiment $e^{\epsilon}_{u,d}$ gives the outcome o^u_1 (respectively o^u_2), when the entity is conditioned for the outcome o^w_2 of the experiment $e^{\epsilon}_{w,c}$ by $P(u, -w, \mu)$ [respectively $P(-u, -w, \mu)$] (see Fig. 6). We repeat again: the conditional probability $P(u, w, \mu)$ is the probability that the experiment $e^{\epsilon}_{u,d}$ gives the outcome o^u_1 if the entity is conditioned on the outcome o^w_1 for the experiment $e^{\epsilon}_{w,c}$. This means that the lack of knowledge on the states is such that if we would decide to perform the experiment $e^{\epsilon}_{w,c}$, the outcome o^w_1 would come out with certainty. In other words (see Fig. 6), the state of the entity is such that the particle is distributed uniformly inside the spherical sector $eig(\{o^w_1\})$, the grey area on Fig. 6. It is easy to formulate in a similar way the other conditional probabilities $P(-u,$

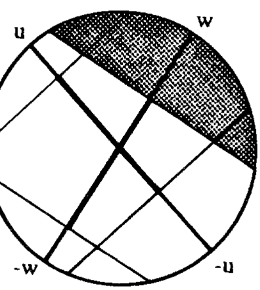


Fig. 6. The situation corresponding to the conditional probability for the ϵ-model. The lack of knowledge about the state of the particle is described by a uniform probability distribution μ on the sphere. For a fixed ϵ and two parameters d and c in the interval $[-1 + \epsilon, 1 - \epsilon]$, we consider two experiments $e^{\epsilon}_{u,d}$ and $e^{\epsilon}_{w,c}$. We want to calculate the conditional probability $P(u, w, \mu)$ that the experiment $e^{\epsilon}_{u,d}$ gives the outcome o^u_1 when the entity is conditioned on the outcome o^w_1 for the experiment $e^{\epsilon}_{w,c}$. This means that before the start of the experiment $e^{\epsilon}_{u,d}$, the situation is such that if we would perform the experiment $e^{\epsilon}_{w,c}$, the outcome o^w_1 would come out with certainty. This conditioning is expressed by a new probability measure on the sphere, which is zero outside the grey area, and equal to the old one, except for a renormalization factor, in the grey area.

w, μ), $P(u, -w, \mu)$, and $P(-u, -w, \mu)$. The explicit calculation of these conditional probabilities is a long exercise of classical calculus, and therefore we refer to Aerts and Aerts (1994b) for a detailed exposition of this calculation. Here we only give the result. Let us call α the angle between the two vectors u and w, then we have

$$P(u, w, \mu) = p_1 \cdot H\left(\epsilon - \cos\frac{\alpha}{2}\right) + H\left(\epsilon - \sin\frac{\alpha}{2}\right) \cdot p_2 \cdot H\left(\cos\frac{\alpha}{2} - \epsilon\right) + p_3 \cdot H\left(\sin\frac{\alpha}{2} - \epsilon\right) \quad (22)$$

where $H(x)$ is the Heaviside function and

$$p_1 = \frac{\cos\alpha(1+\epsilon)}{4\epsilon} + \frac{1}{2} \quad (23)$$

$$p_2 = p_1 + \frac{1}{2} + \frac{\omega(u, w)}{4\pi(1-\epsilon)} + \frac{\cos\alpha + 1}{4\pi\epsilon(1-\epsilon)}\sigma(u, w) \quad (24)$$

$$p_3 = p_1 + \frac{\omega(u, w) - \omega(-u, w)}{4\pi(1-\epsilon)} + \frac{(\cos\alpha - 1)\sigma(-u, w) + (\cos\alpha + 1)\sigma(u, w)}{4\pi\epsilon(1-\epsilon)} \quad (25)$$

where

$$\omega(u, w) = 4\epsilon \operatorname{Arccos}\left(\frac{1 - [\epsilon/\cos(\alpha/2)]^2}{1-\epsilon^2}\right)^{1/2} - 4 \operatorname{Arcsin}\frac{\sin(\alpha/2)}{(1-\epsilon^2)^{1/2}} \quad (26)$$

and

$$\sigma(u, w) = \epsilon \operatorname{tg}\left(\frac{\alpha}{2}\right) \cdot \left\{1 - \left[\frac{\epsilon}{\cos(\alpha/2)}\right]^2\right\}^{1/2} - (1-\epsilon^2) \operatorname{Arccos}\frac{\epsilon \operatorname{tg}(\alpha/2)}{(1-\epsilon^2)^{1/2}} \quad (27)$$

To interpret the graph of this function (see Fig. 7), we first consider the two extremal cases, the classical case, where $\epsilon = 0$, and the quantum case, where $\epsilon = 1$.

1. *The classical case* ($\epsilon = 0$). In this case we find

$$P(u, w, \mu) = 1 - \frac{\alpha}{\pi} \quad (28)$$

which is linear function in the angle (see also Fig. 7 in this respect).

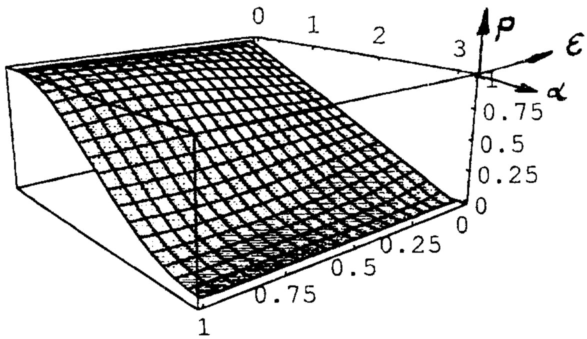


Fig. 7. The conditional probability $P(u, w, \mu)$ for the ϵ-model. It can be seen how this conditional probability evolves continuously from the quantum situation, where $\epsilon = 1$ and it equals the quantum transition probability $\cos^2(\alpha/2)$, where α is the angle between u and w, to the classical situation, where $\epsilon = 0$ and it is a linear function of the angle α between u and w. In some cases of the intermediate situation $0 < \epsilon < 1$ that we shall specify later, the conditional probability cannot be fitted in a Kolmogorovian probability model nor in a quantum probability model.

2. *The quantum case* ($\epsilon = 1$). For this case we only have to take into account the contribution p_1 of (22), and hence we find

$$P(u, w, \mu) = \cos^2(\alpha/2) \tag{29}$$

which is the well-known quantum transition probability between the states p_u and p_w.

Now that we have identified the two extremal cases, we can interpret the graph in Fig. 7, and see that the conditional probability $P(u, w, \mu)$ evolves continuously from the quantum transition probability between the states p_u and p_w to a linear function of the angle between the two vectors u and w.

9. AN INTERMEDIATE SITUATION OF THE ϵ-MODEL THAT IS NEITHER CLASSICAL NOR QUANTUM

A complete probabilistic analysis of the intermediate situation of the ϵ-model is presented in Aerts and Aerts (1994b). Here we only show that for a specific value of $\epsilon = \sqrt{2}/2$ the conditional probabilities of the ϵ-model cannot be fitted into a quantum probability model nor into a classical probability model.

Theorem 4. For $\epsilon = \sqrt{2}/2$, the conditional probabilities of the ϵ-model cannot be fitted into a classical nor into a quantum probability model.

Proof. We shall give a proof *ex absurdum*, and suppose that there does exist a Kolmogorovian model satisfying Bayes' formula for the conditional probability for this value of ϵ. We consider three experiments $e^{\epsilon}_{u,d}$, $e^{\epsilon}_{v,d}$, and

$e^{\epsilon}_{w,d}$, such that $d = 0$ and u, v, and w are in the same plane, with an angle of $\frac{2r}{3}$ between u and v, between v and w, and between w and u. If we use the general expression for the conditional probability (22), we find $P(v, w, \mu) = 0.78$, $P(u, w, \mu) = 0.22$, and $P(-u, v, \mu) = 0.22$. As we have defined, $P(v, w, \mu)$ is the probability that the experiment $e^{\epsilon}_{v,d}$ gives the outcome o^v_1 if we have conditioned the entity in such a way that if we would perform the experiment $e^{\epsilon}_{w,d}$, it would give with certainty an outcome o^w_1. To be able to express more clearly the hypothesis of the existence of a Kolmogorovian probability model, we write these conditional probabilities in a more standard notation. Hence $P(v, w, \mu) = P(e^{\epsilon}_{v,d} = o^v_1 \,|\, e^{\epsilon}_{w,d} = o^w_1)$, $P(u, w, \mu) = P(e^{\epsilon}_{u,d} = o^v_1 \,|\, e^{\epsilon}_{w,d} = o^w_1)$, and $P(-u, v, \mu) = P(e^{\epsilon}_{u,d} = o^v_2 \,|\, e^{\epsilon}_{v,d} = o^v_1)$ under the preparation μ. If there does exist a Kolmogorovian probability model satisfying the Bayes' formula, there exists a set X (playing the role of the sample space), a σ-algebra $\mathcal{B}(X)$ (playing the role of the set of events), and a probability measure ν: $\mathcal{B}(X) \rightarrow [0, 1]$, such that the conditional probabilities $P(e^{\epsilon}_{v,d} = o^v_1 \,|\, e^{\epsilon}_{w,d} = o^w_1)$, $P(e^{\epsilon}_{u,d} = o^v_1 \,|\, e^{\epsilon}_{w,d} = o^w_1)$, and $P(e^{\epsilon}_{u,d} = o^v_2 \,|\, e^{\epsilon}_{v,d} = o^v_1)$ can be written under the appropriate form. This means that there exist elements U, V, $W \in \mathcal{B}(X)$ such that

$$P(e^{\epsilon}_{v,d} = o^v_1 \,|\, e^{\epsilon}_{w,d} = o^w_1) = \frac{\nu(V \cap W)}{\nu(W)} \tag{30}$$

$$P(e^{\epsilon}_{u,d} = o^v_1 \,|\, e^{\epsilon}_{w,d} = o^w_1) = \frac{\nu(U \cap W)}{\nu(W)} \tag{31}$$

$$P(e^{\epsilon}_{u,d} = o^v_2 \,|\, e^{\epsilon}_{v,d} = o^v_1) = \frac{\nu(U^C \cap V)}{\nu(V)} \tag{32}$$

We also have $\nu(W) = \mu(o^w_1, \mu) = 1/2$ and $\nu(V) = \mu(o^v_1, \mu) = 1/2$. Using the fact that ν is a probability measure, and (31) and (32), and substituting the values of the conditional probabilities, we get

$$\nu(V \cap W) = \nu(U \cap V \cap W) + \nu(U^C \cap V \cap W) = \tfrac{1}{2}P(v, w, \mu) = 0.39 \tag{33}$$

$$\nu(U \cap W) = \nu(U \cap V \cap W) + \nu(U \cap V^C \cap W) = \tfrac{1}{2}P(u, w, \mu) = 0.11 \tag{34}$$

$$\nu(U^C \cap V) = \nu(U^C \cap V \cap W) + \nu(U^C \cap V \cap W^C) = \tfrac{1}{2}P(-u, v, \mu) = 0.11 \tag{35}$$

If we subtract (35) from (34), we find

$$\nu(U^C \cap V \cap W) = 0.28 + \nu(U \cap V^C \cap W) \tag{36}$$

which implies that

$$0.28 \leq \nu(U^C \cap V \cap W) \tag{37}$$

On the other hand, from (35) it follows that

$$\nu(U^C \cap V \cap W) \leq 0.11 \tag{38}$$

The inequalities (37) and (38) deliver us the contradiction that we were looking for. The conclusion is that for these values of the conditional probabilities there does not exist a Kolmogorovian probability model satisfying the Bayes formula.

Also for the proof of the nonexistence of a Hilbertian model we proceed *ex absurdum*. If there exists a two-dimensional complex Hilbert space model, where the conditional probabilities are described by the transition probabilities, we can find three orthonormal bases $\{\phi_1, \phi_2\}$, $\{\psi_1, \psi_2\}$, $\{\chi_1, \chi_2\}$ such that

$$\begin{aligned} \gamma^2 &= |\langle \phi_1, \psi_1 \rangle|^2 = |\langle \psi_1, \chi_1 \rangle|^2 = |\langle \chi_1, \phi_2 \rangle|^2 = |\langle \phi_2, \psi_2 \rangle|^2 \\ &= |\langle \psi_2, \chi_2 \rangle|^2 = |\langle \chi_2, \phi_1 \rangle|^2 = 0.78 \end{aligned} \tag{39}$$

$$\begin{aligned} \delta^2 &= |\langle \phi_1, \psi_2 \rangle|^2 = |\langle \psi_1, \chi_2 \rangle|^2 = |\langle \chi_1, \phi_1 \rangle|^2 = |\langle \phi_2, \psi_1 \rangle|^2 \\ &= |\langle \psi_2, \chi_1 \rangle|^2 = |\langle \chi_2, \phi_2 \rangle|^2 = 0.22 \end{aligned} \tag{40}$$

This means that there exist five angles θ_1, θ_2, θ_3, θ_4, and θ_5 such that

$$\begin{aligned} &\langle \chi_1, \psi_2 \rangle = \delta \cdot e^{i\theta_1}, \qquad \langle \chi_1, \phi_1 \rangle = \delta \cdot e^{i\theta_2}, \qquad \langle \phi_1, \psi_2 \rangle = \delta \cdot e^{i\theta_3} \\ &\langle \chi_1, \phi_2 \rangle = \gamma \cdot e^{i\theta_4}, \qquad \langle \phi_2, \psi_2 \rangle = \gamma \cdot e^{i\theta_5} \end{aligned} \tag{41}$$

If $\{\phi_1, \phi_2\}$ is an orthonormal basis, we have $\langle \chi_1, \psi_2 \rangle = \langle \chi_1, \phi_1 \rangle \langle \phi_1, \psi_2 \rangle + \langle \chi_1, \phi_2 \rangle \langle \phi_2, \psi_2 \rangle$, and hence

$$\delta \cdot e^{i\theta_1} = \delta \cdot e^{i\theta_2} \cdot \delta \cdot e^{i\theta_3} + \gamma \cdot e^{i\theta_4} \cdot \gamma \cdot e^{i\theta_5} \tag{42}$$

and also the complex conjugate

$$\delta \cdot e^{-i\theta_1} = \delta \cdot e^{-i\theta_2} \cdot \delta \cdot e^{-i\theta_3} + \gamma \cdot e^{-i\theta_4} \cdot \gamma \cdot e^{-i\theta_5} \tag{43}$$

If we multiply these last equations term by term, we find

$$\delta^2 = \delta^4 + \gamma^4 + \delta^2\gamma^2 e^{i(\theta_2+\theta_3-\theta_4-\theta_5)} + \delta^2\gamma^2 e^{-i(\theta_2+\theta_3-\theta_4-\theta_5)} \tag{44}$$

This we can write as

$$\delta^2 = \delta^4 + \gamma^4 + 2\delta^2\gamma^2 \cos(\theta_2 + \theta_3 - \theta_4 - \theta_5) \tag{45}$$

But then we must have

$$\cos(\theta_2 + \theta_3 - \theta_4 - \theta_5) = \frac{\delta^2 - \delta^4 - \gamma^4}{2\delta^2\gamma^2} \tag{46}$$

If we put in the values $\delta^2 = 0.22$ and $\gamma^2 = 0.78$, we find

$$\cos(\theta_2 + \theta_3 - \theta_4 - \theta_5) = -1.27 \tag{47}$$

From this contradiction we can conclude that there does not exist a two-dimensional Hilbert space model such that the conditional probabilities can be described by transition probabilities in this Hilbert space.

This theorem shows that we really have identified a new region of probabilistic structure in this intermediate domain. In Aerts and Aerts (1994b) we show that for any value of ϵ different from 0, the probability structure of the ϵ-model is non-Kolmogorovian (not satisfying the Bayes formula for the conditional probabilities). We also show that there is a domain of $\epsilon \neq 1$ where a Hilbert space model can be found, but another domain where this is not the case.

We come now to the last section of this paper, where we sketch how these non-Kolmogorovian probabilities appear in other regions of nature.

10. NON-KOLMOGOROVIAN PROBABILITIES IN OTHER REGIONS OF NATURE

As follows from the foregoing analysis, nonclassical experiments, giving rise to nonclassical structure, are characterized by the presence of nonpredetermined outcomes. This makes it rather easy to recognize the nonclassical aspects of experiments in other aspects of reality. Let us consider the situation of a decision process developing in the mind of a human being, and we refer to Aerts and Aerts (1994a) for a more detailed description. Hence our entity is a person, its states being the possible "states" of this person. Experiments are questions that can be asked of the person, to which she or he has to respond with yes or no. The typical situation of an opinion poll can be thought of as a concrete example. Let us consider three different questions:

- *u*: Are you in favor of the use of nuclear energy?
- *v*: Do you think it would be a good idea to legalize soft drugs?
- *w*: Do you think capitalism is better than social-democracy?

We have chosen types of questions about which many persons should not have predetermined opinions. Since the person has to respond with yes or no, she or he, without an opinion before the questioning, needs to form her or his opinion during this process of questioning. We can use the ϵ-model to represent this situation. To simplify the situation, but without touching the essence, we make the following assumptions about the probabilities that are involved. We suppose that in all cases 50% of the persons have answered the question *u* with yes, but only 15% of the persons had a predetermined opinion. This means that 70% of the persons formed their answer during the

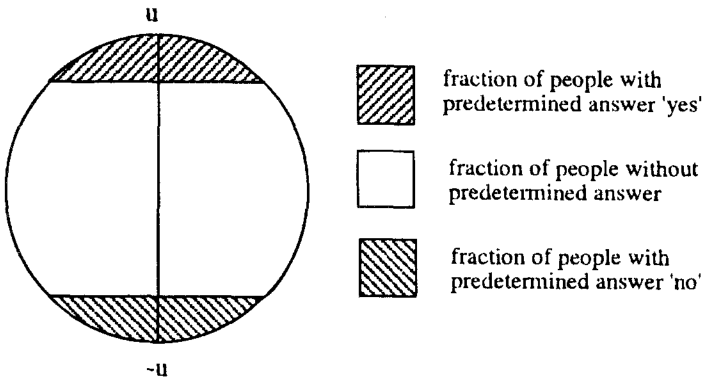
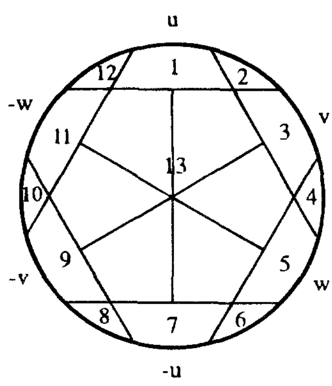


Fig. 8. A representation of the question u by means of the ϵ-model. We have indicated the three regions corresponding to predetermined answer yes, without predetermined answer, and predetermined answer no.

process of questioning. For simplicity we make the same assumptions for v and w. We can represent this situation in the ϵ-model as shown in Fig. 8. We also make some assumptions of the way in which the different opinions related to the three questions influence each other. We can represent an example of a possible interaction by means of the ϵ-model (Fig. 9). One can see how a person can be a strong proponent for the use of nuclear energy, while having no predetermined opinion about the legalization of soft drugs (area 1 in Fig. 9). Area 4 corresponds to a sample of persons who have a predetermined opinion in favor of legalization of soft drugs and in favor of capitalism. For area 10 we have persons who have predetermined opinion against the legalization of soft drugs and against capitalism. All the 13 areas of Fig. 9 can be described in such a simple way.

Fig. 9. A representation of the three questions u, v, and w by means of the ϵ-model. We have numbered the 13 different regions. For example: (1) corresponds to a sample of persons who have predetermined opinion in favor of nuclear energy, but have no predetermined opinion for the other two questions, (4) corresponds to a sample of persons who have predetermined opinion in favor of legalization of soft drugs and in favor of capitalism, (10) corresponds to a sample of persons who have predetermined opinion against the legalization of soft drugs and against capitalism, (13) corresponds to the sample of persons who have predetermined opinion about none of the three questions, etc.



Deliberately we have chosen the different fractions of people in such a way that the conditional probabilities fit into the ϵ-model for a value of $\epsilon = \sqrt{2}/2$. This means that we can apply Theorem 4, and conclude that the collection of conditional probabilities corresponding to these questions u, v, and w can neither be fitted into a Kolmogorovian probability model nor into a quantum probability model. We are developing now in Brussels a statistics for such new situations, which we have called "interactive statistics." By means of this statistics it should be possible to make models for situations where part of the properties to be tested are created during the process of testing.

11. CONCLUSION

The further development of an intermediate (between classical and quantum) probability theory and an interactive statistics could be very fruitful for physics as well as for other sciences.

We are at work now on the construction of a general theory for intermediate structures (Aerts, 1986, Aerts, 1987, Aerts, Durt, and Van Bogaert, 1993, Aerts, 1994, Aerts and Durt, 1994a,b, Coecke, 1995a,b,c). This theory can probably be used to describe the region of reality between microscopic and macroscopic, often referred to as the meso scopic region. Actually, physicists use a very complicated heuristic mixture of quantum and classical theories to construct models for mesoscopic entities. There is, however, no consistent theory, and a general intermediate theory could perhaps fill this gap. We try to find examples of simple physical phenomena in the mesoscopic region that could eventually be modeled by an ϵ-like model (Aerts and Durt, 1994b). If we could succeed in building this intermediate theory, not only we would have a new theory for the mesoscopic region, but the existence of such a theory would also shed light on old problems of quantum mechanics (the quantum–classical relation, the classical limit, the measurement problem, etc.).

So far we have only been developing the kinematics of this intermediate theory (Aerts, 1994; Aerts and Durt, 1994a,b), but once the kinematics is fully developed, the way to construct a dynamics for the intermediate region is straightforward. We can study the imprimitivity system related to the Galilei group and look for representations of this Galilei group in the group of automorphisms of the kinematic structure of the intermediate theory. If we can derive an evolution equation in this way, it should continuously transform with varying ϵ from the Schrödinger equation ($\epsilon = 1$) to the Hamilton equations ($\epsilon = 0$).

As we have shown in the last section of this paper, the development of an interactive statistics could be of great importance for the human sciences, where often nonpredetermined outcome situations appear. It could lead to a new methodology for these sciences. Actually, one is aware of the problem of the interaction between subject and object, but it is generally thought that this problem cannot be taken into account in the theory.

We also want to remark that the "hidden measurement approach" defines a new quantization procedure. Starting from a classical mechanical entity and adding "lack of knowledge (or fluctuations)" on the measurement process, out of the classical entity appears a quantumlike entity. Another problem that we are investigating is an attempt to describe quantum chaos by means of this new quantization procedure. It can be shown that the sensitive dependence on the initial conditions that can be found in the ϵ-model for the classical situation in the set of unstable equilibrium states disappears when the fluctuations on the measurement process increase. This could be the explanation for the absence of quantum chaos.

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REFERENCES

Accardi, L. (1982). On the statistical meaning of the complex numbers in quantum mechanics, *Nuovo Cimento*, **34**, 161.

Aerts, D. (1986). A possible explanation for the probabilities of quantum mechanics, *Journal of Mathematical Physics*, **27**, 202.

Aerts, D. (1987). The origin of the non-classical character of the quantum probability model, in *Information, Complexity, and Control in Quantum Physics*, A. Blanquiere, *et al.*, eds., Springer-Verlag, Berlin.

Aerts, D. (1991). A macroscopic classical laboratory situation with only macroscopic classical entities giving rise to a quantum mechanical probability model, in *Quantum Probability and Related Topics*, Vol. VI, L. Accardi, ed., World Scientific, Singapore.

Aerts, D. (1994). Quantum structures, separated physical entities and probability, *Foundations of Physics*, **24**, 1227.

Aerts, D., and Aerts, S. (1994a). Applications of quantum statistics on psychological studies of decision processes, *Foundations of Science*, **1**, 85.

Aerts, D., and Aerts, S. (1994b). Interactive probability models: From quantum in Kolmogorovian, preprint, TENA, VUB, Brussels, Belgium.

Aerts, D., and Durt, T. (1994a). Quantum, classical and intermediate, an illustrative example, *Foundations of Physics*, **24**, 1353.

Aerts, D., and Durt, T. (1994b). Quantum, classical and intermediate; a measurement model, in *Symposium on the Foundations of Modern Physics*, Helsinki 1994, ed. K. V. Laurikainen, C. Montonen, K. Sunnar Borg, Editions Frontières, Gives Sur Yvettes, France.

Aerts, D., and Van Bogaert, B. (1992). A mechanical classical laboratory situation with a quantum logic structure, *International Journal of Theoretical Physics*, **10**, 1893.

Aerts, D., Durt, T., and Van Bogaert, B. (1992). A physical example of quantum fuzzy sets, and the classical limit, in Proceedings of the First International Conference on Fuzzy Sets and their Applications, *Tatra Mountains Mathematical Publications,* **1**, 5.

Aerts, D., Durt, T., and van Bogaert, B. (1993a). Quantum probability, the classical limit and non-locality, in *Symposium on the Foundations of Modern Physics,* T. Hyvonen, ed., World Scientific, Singapore.

Aerts, D., Durt, T., Grib, A. A., Van Bogaert, B., and Zapatrin, R. R. (1993b). Quantum structures in macroscopic reality, *International Journal of Theoretical Physics,* **32**, 3, 489.

Birkhoff, G., and von Neumann, J. (1936). *Annals of Mathematics,* **37**, 823.

Coecke, B. (1995a). A hidden measurement model for pure and mixed states of quantum mechanics in Euclidean Space, *Int. J. Theor. Phys.*, this issue.

Coecke, B. (1995b). Hidden measurement representation for quantum entities described by finite dimensional complex Hilbert spaces, to be published in *Foundations of Physics*, 1995.

Coecke, B., (1995c). Generalization of the proof on the existence of hidden measurements to experiments with an infinite set of outcomes, to be published in *Foundations of Physics Letters*, 1995.

Emch, G. G. (1984). *Mathematical and Conceptual Foundations of 20th Century Physics,* North-Holland, Amsterdam.

Foulis, D., and Randall, C. (1972). *Journal of Mathematical Physics,* **1972**, 1667.

Gudder, S. P. (1988). *Quantum Probability,* Academic Press, New York.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics,* Addison-Wesley, Reading, Massachusetts.

Piron, C. (1976). *Foundations of Quantum Physics,* Benjamin, New York.

Pitovski, I. (1989). *Quantum Probability–Quantum Logic,* Springer-Verlag, Berlin.

Randall, C., and Foulis, D. (1979). The operational approach to quantum mechanics, in *Physical Theory as Logico-Operational Structure,* C. A. Hooker, ed., Reidel, Dordrecht.

Randall, C., and Foulis, D. (1983). A mathematical language for quantum physics, in *Les Fondements de la Mécanique Quantique,* C. Gruber *et al.*, eds., AVCP, Lausanne, Switzerland.

Segal, I. E. (1947). *Annals of Mathematics,* **48**, 930.